On Zermelo's Paper "On the Mechanical Explanation of Irreversible Processes" *,†

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SUMMARY

The second law of thermodynamics can be proved from the mechanical theory if one assumes that the present state of the universe, or at least that part which surrounds us, started to evolve from an improbable state and is still in a relatively improbable state. This is a reasonable assumption to make, since it enables us to explain the facts of experience, and one should not expect to be able to deduce it from anything more fundamental.

The applicability of probability theory to physical situations, which is disputed by Zermelo, cannot by rigorously proved, but the fact that one never observes those events that theoretically should be quite rare is certainly not a valid argument against the theory.

One may speculate that the universe as a whole is in thermal equilibrium and therefore dead, but there will be local deviations from equilibrium which may last for the relatively short time of a few eons. For the universe as a whole, there is no distinction between the "backwards" and "forwards" directions of time, but for the worlds on which living beings exist, and which are therefore in relatively improbable states, the direction of time will be determined by the direction of increasing entropy, proceeding from less to more probable states.

I will be as brief as possible without loss of clarity.

§1. The second law will be explained mechanically by means of an assumption A (which is of course unprovable) that the universe,

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considered as a mechanical system—or at least a very large part of it which surrounds us—started from a very improbable state, and is still in an improbable state. Hence, if one takes a smaller system of bodies in the state in which he actually finds them, and suddenly isolates this system from the rest of the world, then the system will initially be in an improbable state, and as long as the system remains isolated it will always proceed toward more probable states. On the other hand, there is a very small probability that the enclosed system is initially in thermal equilibrium, and that while it remains enclosed it moves far enough away from equilibrium that its entropy decrease is noticeable.

The question is not what will be the behaviour of a completely arbitrary system, but rather what will happen to a system existing in the present state of the world. The initial state precedes the later states, so that Zermelo’s conclusion that all points of the $H$-curve must be maxima is invalid. Hence, it turns out that entropy always increases, temperature and concentration differences are always equalized, that the initial value of $H$ is such that during the time of observation it almost always decreases, and that initial and final states are not interchangeable, in contradiction to Zermelo’s assertions. Assumption $A$ is a comprehensible physical explanation of the peculiarity of the initial state, consistent with the laws of mechanics; or better, it is a unified viewpoint corresponding to these laws, which allows one to predict the type of peculiarity of the initial state in any special case; for one can never expect that the explanatory principle must itself be explained.

On the other hand, if we do not make any assumption about the present state of the universe, then of course we cannot expect to find that a system isolated from the universe, whose initial state is completely arbitrary, will be in an improbable state initially rather than later. On the contrary it is to be expected that at the moment of separation the system will be in thermal equilibrium. In the few cases where this does not happen, it will almost always be found that if the state of the isolated system is followed either backwards or forwards in time, it will almost immediately pass to a more
probable state. Much rarer will be the cases in which the state becomes still more improbable as time goes on; but such cases will be just as frequent as those where the state becomes more improbable as one follows it backwards in time.

§2. The applicability of probability theory to a particular case cannot of course be proved rigorously. If, out of 100,000 objects of a certain kind, about 100 are annually destroyed by fire, then we cannot be sure that this will happen next year. On the contrary, if the same conditions could be maintained for $10^{10}$ years, then during this time it would often happen that all 100,000 objects would burn up on the same day; and likewise there will be entire years during which not a single object is damaged. Despite this, every insurance company relies on probability theory.

It is even more valid, on account of the huge number of molecules in a cubic millimetre, to adopt the assumption (which cannot be proved mathematically for any particular case) that when two gases of different kinds or at different temperatures are brought in contact, each molecule will have all the possible different states corresponding to the laws of probability and determined by the average values at the place in question, during a long period of time. These probability arguments cannot replace a direct analysis of the motion of each molecule; yet if one starts with a variety of initial conditions, all corresponding to the same average values (and therefore equivalent from the viewpoint of observation), one is entitled to expect that the results of both methods will agree, aside from some individual exceptions which will be even rarer than in the above example of 100,000 objects all burning on the same day. The assumption that these rare cases are not observed in nature is not strictly provable (nor is the entire mechanical picture itself) but in view of what has been said it is so natural and obvious, and so much in agreement with all experience with probabilities, from the method of least squares to the dice game, that any doubt on this point certainly cannot put in question the validity of the theory when it is otherwise so useful.

It is completely incomprehensible to me how anyone can see a refutation of the applicability of probability theory in the fact that
some other argument shows that exceptions must occur now and then over a period of eons of time; for probability theory itself teaches just the same thing.

§3. Let us imagine that a partition which separates two spaces filled with different kinds of gas is suddenly removed. One could hardly find another situation (at least one in which the method of least squares is applicable) where there are so many independent causes acting in such different ways, and in which the application of probability theory is so amply justified. The opinion that the laws of probability are not valid here, and that in most cases the molecules do not diffuse, but instead a large part of the container has significantly more nitrogen, and another part has significantly more oxygen, cannot be disproved, even if I were to calculate exactly the motions of trillions of molecules in millions of different special cases. Nevertheless this opinion certainly does not have enough justification to cast doubt on the usefulness of a theory that starts from the assumption of the applicability of probability theory and draws the logical consequence from this assumption.

Poincaré's theorem does not contradict the applicability of probability theory but rather supports it, since it shows that in eons of time there will occur a relatively short period during which the state probability and the entropy of the gas will significantly decrease, and that a more ordered state similar to the initial state will occur. During the enormously long period of time before this happens, any noticeable deviation of the entropy from its maximum value is of course very improbable; however, a momentary increase or decrease of entropy is equally probable.

It is also clear from this example that the process goes on irreversibly during observable times, since one intentionally starts from a very improbable state. In the case of natural processes this is explained by the assumption that one isolates the system of bodies from the universe which is at that time in a very improbable state as a whole.

This example of two initially unmixed gases gives us incidentally a possible way of imagining the initial state of the world. For if in the example we isolate the gas found in a smaller space soon after
the beginning of the diffusion from the rest of the gas, we will have
the asymmetry with respect to forward and backward steps in time
as in the isolated system of bodies mentioned in §1.

§4. I myself have repeatedly warned against placing too much
confidence in the extension of our thought pictures beyond the
domain of experience, and I am aware that one must consider the
form of mechanics, and especially the representation of the
smallest particles of bodies as mass-points, to be only provi-
sionally established. With all these reservations, it is still possible
for those who wish to give in to their natural impulses to make up a
special picture of the universe.

One has the choice of two kinds of pictures. One can assume
that the entire universe finds itself at present in a very improbable
state. However, one may suppose that the eons during which this
improbable state lasts, and the distance from here to Sirius, are
minute compared to the age and size of the universe. There must
then be in the universe, which is in thermal equilibrium as a whole
and therefore dead, here and there relatively small regions of the
size of our galaxy (which we call worlds), which during the
relatively short time of eons deviate significantly from thermal
equilibrium. Among these worlds the state probability increases
as often as it decreases. For the universe as a whole the two
directions of time are indistinguishable, just as in space there
is no up or down. However, just as at a certain place on the earth’s
surface we can call “down” the direction toward the centre of the
earth, so a living being that finds itself in such a world at a certain
period of time can define the time direction as going from less
probable to more probable states (the former will be the “past”
and the latter the “future”) and by virtue of this definition he
will find that this small region, isolated from the rest of the
universe, is “initially” always in an improbable state. This
viewpoint seems to me to be the only way in which one can under-
stand the validity of the second law and the heat death of each
individual world without invoking an unidirectional change of the
total universe from a definite initial state to a final state. The
objection that it is uneconomical and hence senseless to imagine
such a large part of the universe as being dead in order to explain why a small part is living—this objection I consider invalid. I remember only too well a person who absolutely refused to believe that the sun could be 20 million miles from the earth, on the grounds that it is inconceivable that there could be so much space filled only with aether and so little with life.

§5. Whether one wishes to indulge in such speculations is of course a matter of taste. It is not a question of choosing as a matter of taste between the Carnot–Clausius principle and the mechanical theory. The importance of the former, as the simplest expression of the facts so far observed, is not in dispute. I assert only that the mechanical picture agrees with it in all actual observations. That it suggests the possibility of certain new observations—for example, of the motion of small particles in liquids and gases, and of viscosity and heat conduction in very rarefied gases, etc.—and that it does not agree with the Carnot–Clausius principle on some unobservable questions (for example the behaviour of the universe or a completely enclosed system during an infinite period of time), may be called a difference in principle, if you like. In any case it provides no basis for giving up the mechanical theory, as Herr Zermelo would like to do, if it cannot be changed in principle (which one should not expect). It is precisely this difference that seems to me to indicate that the universality of our thought-pictures will be improved by studying not only the consequences of the principle in the Carnot–Clausius version but also in the mechanical version.

Appendix

§6. I have always measured the probability of a state, independently of its temporal duration, by the "extension $\gamma$" (as Zermelo calls it) of its corresponding region, and I used the Liouville theorem in this connection 30 years ago.† The Maxwellian state

† See especially Wien. Ber. 58, 517 (1868); 63, 679, 712 (1871); 66, (1872); 76, 373 (1877). I have given there the proof of the above-mentioned theorem, which there is not space to repeat here.
is simply the most probable because it can be realized in the largest number of ways. The total extension $\gamma$ of the region of all those states for which the velocity distribution is approximately given by the Maxwell distribution is therefore much greater than the total extension of the regions of all other states. It was only to illustrate the relation between the temporal course of the states and their probabilities that I represented the reciprocal value of this probability for the different successive states by the $H$-curve, in the case of a large finite number of hard gas molecules. Aside from a vanishingly small number of special initial states, the most probable states will also occur the most frequently (at least for a very large number of molecules). The ordinates of this curve are almost always very small, and these small ordinates are of course not usually maxima. It is only the ordinates with unusually large values that are mostly maxima, and indeed they are more likely to be maxima the greater they are. The fact that a very large ordinate $H_0$ is more often a maximum than the intersection of the line $y = H_0$ with a still higher peak is a consequence of the enormous increase in rarity of peaks with increasing height. See the above figure, which is of course to be taken with a pinch of salt. A correct figure could not be printed because the $H$-curve actually has a large number of maxima and minima within each finite segment, and cannot be represented by a line with continuously changing direc-
tion. It would be better to call it an aggregate of many points very close together, or small horizontal segments.†

The Poincaré theorem is of course inapplicable to a terrestrial body which we can observe, since such a body is not completely isolated; likewise, it is inapplicable to the completely isolated gas treated by the kinetic theory, if one first lets the number of molecules become infinite, and then the quotient of the time between successive collisions and the time of observation.

Vienna, 16 December 1896.

† Nature 51, 413, 581 (1895).